Determine if each of the following converges or diverges.
If it converges, write "CONVERGES". If it diverges, write "DIVERGES".
Justify each answer using proper mathematical reasoning, algebra and/or calculus.

ALL HIGHLIGHTED ITEMS WORTH ① POINT UNLESS OTHERWISE NOTED

[a] $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ SCORE: ____ / 7 PTS

$\frac{1}{x(\ln x)^3}$ IS CONTINUOUS ON $[2,\infty)$ $\frac{1}{2}$

$\frac{1}{2}$ >0 SINCE $x, \ln x > 0$ ON $[2,\infty)$

AND DECREASING ON $[2,\infty)$

SINCE $\frac{d}{dx}\frac{1}{x(\ln x)^3} = \frac{-((\ln x)^3 + 3(\ln x)^2)}{x^2(\ln x)^6} < 0$ ON $[2,\infty)$

$$\int_2^\infty \frac{1}{x(\ln x)^3}\,dx = \int_{\ln 2}^\infty \frac{1}{u^3}\,du = \lim_{N\to\infty}\left(-\frac{1}{2u^2}\right)\Big|_{\ln 2}^{N} \text{ OR } \lim_{x\to\infty}\left(-\frac{1}{2(\ln x)^2}\right)\Big|_2^\infty$$

$u = \ln x$
$du = \frac{1}{x}dx$

$$= \lim_{N\to\infty} -\frac{1}{2}\left(\frac{1}{N^2} - \frac{1}{(\ln 2)^2}\right) = \frac{1}{2(\ln 2)^2}$$

$$\text{OR } \lim_{x\to\infty} -\frac{1}{2}\left(\frac{1}{(\ln x)^2} - \frac{1}{(\ln 2)^2}\right)$$

CONVERGES

SO $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ CONVERGES BY INTEGRAL TEST

[b] $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^8 + 84n^2 + 506}}{(8n^2 + 4n + 1)^2}$ SCORE: ____ / 8 PTS

LET $b_n = \frac{\sqrt[3]{n^8}}{(8n^2)^2} = \frac{1}{64}\cdot\frac{1}{n^{4/3}}$

$$\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt[3]{n^8+84n^2+506}}{(8n^2+4n+1)^2}\cdot\frac{(8n^2)^2}{\sqrt[3]{n^8}}\right|$$

$$= \lim_{n\to\infty}\left|\sqrt[3]{\frac{n^8+84n^2+506}{n^8}}\left(\frac{8n^2}{8n^2+4n+1}\right)^2\right|$$

$$= \sqrt[3]{1}\cdot 1^2 = 1 \neq 0$$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{64}\cdot\frac{1}{n^{4/3}}$ CONVERGES $(p = \frac{4}{3} > 1)$ BY p-SERIES TEST

SO $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^8+84n^2+506}}{(8n^2+4n+1)^2}$ CONVERGES BY LIMIT COMPARISON TEST

[c] $\sum_{n=1}^{\infty} \frac{200n-5}{n+n^2 2^n}$ SCORE: _____ / $5\frac{1}{2}$ PTS

$\frac{1}{2}$ $$\underbrace{0 < \frac{200n-5}{n+n^2 2^n}} < \frac{200n}{n^2 2^n} = \frac{200}{n2^n} < \underbrace{\frac{200}{2^n}}$$

$\sum_{n=1}^{\infty} \frac{200}{2^n}$ CONVERGES ($|r| = \frac{1}{2} < 1$) BY GEOMETRIC SERIES TEST

SO $\sum_{n=1}^{\infty} \frac{200n-5}{n+n^2 2^n}$ CONVERGES BY COMPARISON TEST

[d] $\sum_{n=1}^{\infty} \frac{e^{-\frac{1}{n}}}{\tan^{-1} n}$ SCORE: _____ / 4 PTS

$$\lim_{n\to\infty} \frac{e^{-\frac{1}{n}}}{\tan^{-1} n} = \frac{e^0}{\frac{\pi}{2}} = \frac{2}{\pi} \neq 0$$

$\sum_{n=1}^{\infty} \frac{e^{-\frac{1}{n}}}{\tan^{-1} n}$ DIVERGES BY DIVERGENCE TEST

[e] $\sum_{n=1}^{\infty} \frac{4^n}{n!}$ SCORE: _____ / $5\frac{1}{2}$ PTS

$\frac{1}{2}$ $$0 < \frac{4^n}{n!} = \frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdots 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots n}$$

$$< \frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdots 4}{1\ 2\ 3\ 4\ 5\ 5 \cdots 5} = \frac{4^4}{4!}\left(\frac{4}{5}\right)^{n-4}$$

$\sum_{n=1}^{\infty} \frac{4^4}{4!}\left(\frac{4}{5}\right)^{n-4}$ CONVERGES ($|r| = \frac{4}{5} < 1$) BY GEOMETRIC SERIES TEST

SO $\sum_{n=1}^{\infty} \frac{4^n}{n!}$ CONVERGES BY COMPARISON TEST